UNFOLDINGS

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ABSTRACT

It's almost a slang term, but there are many ideas of unfoldings in Mathematics. Perhaps the most straightforward is the topological idea of a universal cover, but there are discrete versions as well, and they crop up in Possible World Semantics, Formal Language Theory, BQO Theory and in Randall Holmes' proof of the consistency of Quine's NF. In this introductory survey i shall try to tie these various ideas together and see what the core ideas are.

Cayley graphs
Article [1] by Fürer and Specker on paying attention to history.
Regular languages
Sahlqvist on frames [3]
BQO theory: derivatives of blocks
Universal covers of graphs
Covering spaces
Coverings of games as in the proof of Borel Determinacy
Holmes' proof of Con(NF)—[2]

References

[1] Fürer, M., Specker, E.: "Learning from history can help (preliminary version)". Abstract of a paper presented to the meeting of the ASL, New York City, May 1987, In: *Journal of Symbolic Logic* **53** (1988), p. 1273

[2] `https://randall-holmes.github.io/Nfproof/newattempt.pdf`

[3] Henrik Sahlqvist, "Completeness and Correspondence in the First and Second Order Semantics for Modal Logic", in Proceedings of the Third Scandinavian Logic Symposium, 1975, pp 110–143, S. Kanger ed, North-Holland.

A model of TTT is a linearly ordered indexed family of sets $\{A_i : i \in I\}$ with the property that whenever $i_1 < i_2 < i_3 \cdots$ is an increasing sequence of indices then we can find binary relations $\in_{n,n+1} \subseteq A_{i_n} \times A_{i_{n+1}}$ with which we can equip that sequence of A_i so that they become a model of simply typed set theory.

Definition 1

- *The* **derivative** *(version one) of a binary structure $\langle X, R\rangle$ is a ternary structure with the same carrier set and the relation:*

$$\{\langle x, y, z\rangle : R(x, y) \wedge R(y, z)\}$$

- *The* **derivative** *(version two) of a binary structure $\langle X, R\rangle$ is a binary structure with carrier set R and the relation*

$$\{\langle\langle x, y\rangle, \langle y, z\rangle\rangle : \langle x, y\rangle,\ \langle y, z\rangle \in R\}$$

... and we write them both $D(\langle X, R\rangle)$.

Version one has the same carrier set but different arity;
Version two has the same arity but different carrier set.

Definition 2 *The (version one) derivative $D(\langle X, R\rangle)$ of an n-ary structure $\langle X, R\rangle$ is an $(n + 1)$-ary structure with the same carrier set X but with relation*

$$\{\langle x_0, \cdots, x_n\rangle : R(x_0, \cdots, x_{n-1}) \wedge R(x_1, \cdots, x_n)\}$$

All the $D^n(\langle X, R\rangle)$ have the same carrier set. This means we can form the structure

Definition 3

$$\langle X, \bigcup_{i\in\mathbb{N}} D^n(R)\rangle$$

which we then think of as the **unfolding** *of $\langle X, R\rangle$.*

Application to BQO theory

For S a binary relation on Y, $\langle \mathcal{P}(Y), S^+\rangle$ is the structure with carrier set $\mathcal{P}(Y)$ and S^+ defined by $S^+(A, B)$ iff $(\forall y \in A)(\exists y' \in B)(S(y, y'))$.

Let $\langle X, R\rangle$ and $\langle Y, S\rangle$ be two binary structures.
Consider the following relation between them:

$$(\forall f : X \to Y)(\exists x \neq x' \in X)(R(x, x') \wedge S(f(x), f(x'))$$

(The paradigmatic example of this situation is where $\langle X, R\rangle$ is $\langle \mathbb{N}, \leq_{\mathbb{N}}\rangle$ and $\langle Y, S\rangle$ is a quasiorder. In these circumstances we say $\langle Y, S\rangle$ is a *wellquasiorder* aka WQO.)
Then

Remark 1 *$\langle X, R\rangle$ is related to $\langle \mathcal{P}(Y), S^+\rangle$ iff $D(\langle X, R\rangle)$ is related to $\langle Y, S\rangle$.*

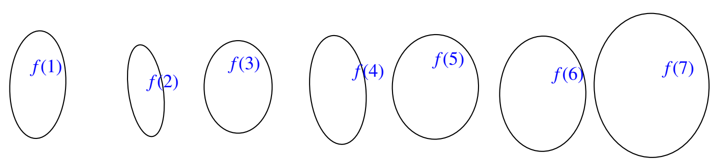
Here $D(\langle X, R\rangle)$ is of course the binary version not the ternary version.
Proof:
We have to prove a biconditional `Left` $\longleftrightarrow$ `Right`, where
`Left` is "$\langle X, R\rangle$ is related to $\langle \mathcal{P}(Y), S^+\rangle$" and
`Right` is "$D(\langle X, R\rangle)$ is related to $\langle Y, S\rangle$".

$\neg$`Left` $\to$ $\neg$`Right`

Suppose $f : X \to \mathcal{P}(Y)$ is a counterexample to

$$(\forall f : X \to Y)(\exists x \neq x' \in X)(R(x, x') \wedge S^+(f(x), f(x'))$$

Then, for all $x \neq x' \in X$ with $R(x, x')$, we have $\neg S^+(f(x), f(x')$.



So $\neg(\forall y \in f(x))(\exists y' \in f(x'))S(y, y')$, which is to say $(\exists y \in f(x))(\forall y' \in f(x'))\neg S(y, y')$.
For each such pair $x \neq x'$ pick such a y—call it y_0—and define $g(\langle x, x'\rangle) = y_0$.
This g is now a counterexample to `Right`.

$\neg$`Right` $\to$ $\neg$`Left`

Given g a counterexample to `Right` set $f(x) = \{g(x, x') : x' \in X\}$. f is now a counterexample to `Left`.

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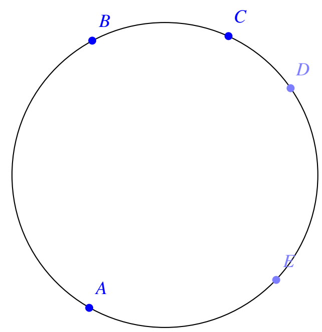
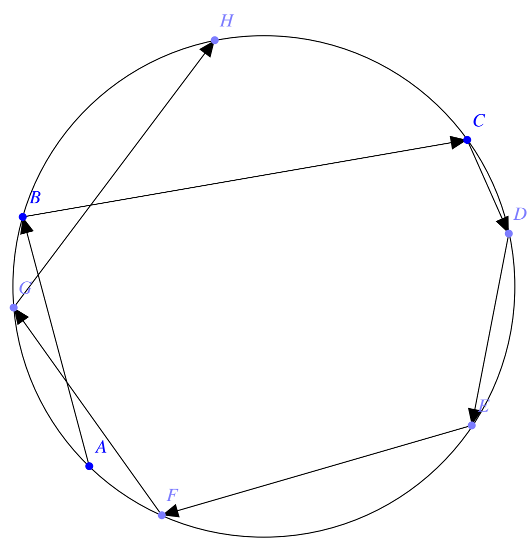


$R(A,B,C), R(B,C,D), R(D,E,A), R(E,A,B)$



$< ABCDEFG >$ has loop number 1